

Rain, power laws, and advection

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Abstract

Localized rain events have been found to follow power-law size and duration distributions over several decades, suggesting parallels between precipitation and seismic activity [O. Peters et al., PRL **88**, 018701 (2002)]. Similar power laws are generated by treating rain as a passive tracer undergoing advection in a velocity field generated by a two-dimensional system of point vortices.

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As natural phenomena are probed on ever finer length and time scales, surprising scale-invariant properties are brought to light. A notable instance is the recent discovery that distributions of rainy and dry intervals, and the size of rain events, follow power laws [1,2]. Analyzing Doppler radar data, Peters, Hertlein and Christensen found that the distribution of local rain-event sizes decays as a power law over at least three decades. Durations of rain-free intervals (‘droughts’) are also power-law distributed over the range of several minutes to about a week, with a significant perturbation reflecting diurnal variation.

The similarity between these observations and scaling laws in seismic activity (the Gutenberg-Richter law for earthquake sizes, and the Omori law for waiting times) suggests a parallel between atmospheric precipitation and relaxation of the Earth’s crust at stressed tectonic-plate boundaries [2]. In the latter context, cooperative relaxation due to elastic interactions and nonlinear friction is captured by block-spring models [3,4] or, in much-reduced fashion, by sandpile models [5]. But if certain aspects of rain distributions resemble those of avalanches in sandpile-like models, the underlying physics remains obscure. There is no obvious reason for the formation or precipitation of one raindrop to provoke similar events nearby. Given the attendant release of latent heat, one would instead expect a self-limiting tendency in condensation. Thus it appears more promising to seek the explanation for power-law distributions in atmospheric dynamics.

Atmospheric motion involves turbulence, particularly in the vicinity of storms; various aspects of turbulent flow follow power laws over many orders of magnitude [6,7]. Even in the absence of fully developed turbulence, unsteady flow will stretch and fold any initially compact region, leading to a highly convoluted, nonuniform density of suspended particles or droplets [8,9] via chaotic advection [10]. In light of these observations, it is interesting to develop a model in which rain is an *ideal passive tracer*, following the local fluid velocity $\mathbf{u}(\mathbf{x}(t), t)$ [11,12]. ($\mathbf{x}(t)$ denotes the instantaneous position of the tracer.) This Letter aims to show that such a model is capable of producing power-law-distributed event sizes and durations.

I adopt a simplified model susceptible to numerical simulation. A convenient alternative to numerical integration of the Navier-Stokes equation is afforded by vortex dynamics [6]. In the two-dimensional case studied here, a system of N_V point vortices [13], each moving in the velocity field generated by all vortices except itself, is a particularly attractive method for simulating incompressible, inviscid flow. Point-vortex systems have been used for some time in studies of two-dimensional turbulence [6,14,15].

The simulation cell is a unit square with periodic boundaries. The velocity of vortex i is given by

$$\mathbf{v}_i = \sum_{j \neq i} \frac{K_j}{2\pi r_{ij}^2} \hat{\mathbf{k}} \times \mathbf{r}_{ij}, \quad (1)$$

where K_j represents the circulation of vortex j (equal numbers of clockwise and anticlockwise vortices are used), and $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, under period boundaries, using the nearest-image criterion. The velocity $\mathbf{u}(\mathbf{x}, t)$ at an arbitrary point \mathbf{x} in the plane (not occupied by a vortex) is given by a similar sum including contributions from

all vortices. The number of vortices N_V ranges from 10 to 126.

I study several types of vortex-strength distributions. The simplest assigns all vortices the same strength $|K|$. Other studies employ a hierarchical vortex distribution, defined as follows. The zeroth “generation” consists of a pair of vortices with $K = \pm K_0$. Subsequent generations, $n = 1, \dots, g$ have 2^{n+1} vortices, with circulation $|K| = K_0/\alpha^n$. I study $\alpha = 2, 3$, and 4 , using $g+1 = 5$ or 6 generations. The purpose of the hierarchical distribution is to provide structure on a variety of length scales, without trying to reproduce any specific energy spectrum $\epsilon(k)$. The vortices are assigned random initial positions, but their subsequent evolution is deterministic. Being point objects, they possess no intrinsic length scale. (Note however that in the presence of other vortices, the ‘sphere of influence’ of vortex i is proportional to K_i .) A characteristic length scale is the mean separation $\sim 1/\sqrt{N_V}$ between vortices. The vortex system defines a mean speed $u = \langle |\mathbf{u}(\mathbf{x}, t)| \rangle \propto K_0\sqrt{N_V}$; an important time scale is $\tau_C \sim 1/u$, the typical time for a fluid particle to traverse the system.

A large number of tracers, $N_p = 5\,000$ or $10\,000$, are thrown at random into a small region (of linear dimension 0.05) to simulate a localized condensation event. (Alternatively, the tracer-laden region may be interpreted as a parcel of atmosphere of high humidity, destined to generate precipitation.) ‘Rain’ corresponds to the presence of one or more tracers in a very small predefined region or ‘weather station’, of linear dimension 0.01 . (To improve statistics, I use $40 - 50$ such stations; each is randomly assigned a fixed position.) At each step of the integration ($\Delta t = 5 \times 10^{-6} - 10^{-4}$), the number of particles $n_i(t)$ at each station i is monitored. A sequence of nonzero occupation numbers at a given station constitutes a rain event, just as in the radar observations [1]; the size of a rain event is $s = \sum_t n_i(t)$ where the sum is over the set of consecutive time steps for which $n_i(t) > 0$. In case $n_i = 0$, station i is said to experience a drought. The durations of droughts and of rain events are likewise monitored over a time interval T .

As anticipated, the highly irregular velocity field stretches and folds the tracer-bearing region [10]. At certain hyperbolic points the flow bifurcates. Fig. 1 shows the particle positions at a time $\sim 1.3\tau_C$, in a system with 126 vortices ($\alpha = 3$). The tracers are widely scattered, but their distribution remains highly nonuniform. At later times ($\sim 50\tau_C$), tracers are to be found throughout the system, but there are empty regions of various sizes centered on vortices or vortex pairs.

The nonuniform tracer density leads to scale-invariant rain and drought distributions. Fig. 2 shows the number of rain events $N(s)$ as a function of size s , in a study with 126 vortices, $\alpha = 2$, and $T \simeq \tau_C/2$ (the mean velocity $u \simeq 5$; statistics are accumulated over a total of 200 realizations). The distribution follows a power law, $N(s) \sim s^{-0.95(1)}$ over nearly five decades. (Figures in parentheses denote uncertainties.) Fig. 3 shows the drought-duration distribution for the same parameters; it follows a power law, with a decay exponent of $1.14(1)$, over about three decades. The distribution of rain *durations* (Fig. 3, inset) is more complicated, decaying first as a power law (with an exponent of about 1), over a decade or so, then attaining a roughly constant value, and finally decaying exponentially.

Note that for durations larger than $T/\Delta t = 10^4$, the drought distribution decays

exponentially, as expected for independent events. The largest possible rain size is $N_p T / \Delta t = 10^8$, corresponding to *all* of the tracers sitting at a particular station for an entire simulation. The largest rain event observed was 2.5×10^6 , or 2.5% the maximum possible. Given the small area of a station, this represents a remarkable concentration of tracers for a long time, reminiscent of blocking effects in the atmosphere, that may cause a coherent structure (possibly a vortex-dipole) to remain stationary over a long interval [16].

Varying the vortex distribution and observation interval T , the following trends emerge. For T/τ_C in the range 0.1 - 2, a power-law rain-size distribution, with an exponent in the range of 0.93 - 1.02, is observed over 4 - 5 1/2 decades. The drought-duration distribution decays with a somewhat larger exponent, 1.12 - 1.16, and follows a power law over 3 - 4 decades. Larger exponent values are associated with higher values of α ; these yield somewhat smaller ranges for the power laws. Conversely, the largest power-law range, and smallest exponent values, are observed when all vortices are of equal strength. There is no significant difference between the distributions obtained initially and those found after the vortices have had some time to evolve, suggesting that the equilibration process expected in two-dimensional turbulence [15,14] is not important as regards rain and drought statistics [17].

Even systems with as few as ten vortices yield good power laws; examples are shown in Fig. 4. This indicates that chaotic advection is the essential feature leading to scale invariance, rather than well developed turbulence.

For larger values of T/τ_C the particles are more dispersed, and the rain size and drought duration follow stretched-exponential functions; an example is shown in Fig. 2 (inset) for $T \simeq 200\tau_C$. Even for large values of T/τ_C (up to 200 in the present study), the distributions decay more slowly than an exponential, showing that the tracer density is non-Poissonian.

The simulation results may be summarized as showing scale-invariant rain-size and drought-duration distributions for intervals such that the tracers remain highly clustered. (Taking the scale of the original rain-formation region as ~ 1 km, the scale-invariant distributions correspond to rain scattered over a region of $\sim 10 - 100$ km. If we adopt a typical wind velocity $u \sim 20$ km/h then the simulation time-step Δt is on the order of one second.) Although the decay exponents are somewhat smaller than those obtained from observational data (1.36 and 1.42 for rain size and drought duration, resp. [1]), the simulations also show the drought duration decaying more rapidly than that for rain event sizes. For conditions under which the rain is more thoroughly dispersed, simulations yield stretched-exponential distributions. While the latter have not been reported, it is well to recall that the observational data come from a single station. Observations from other sites are needed to confirm the generality of power laws and the possibility of other (non-scale-invariant) forms.

Clearly, the model employed in the present “proof-of-principle” study contains a minimum of atmospheric physics. A three-dimensional description, allowing for stratification, convection, and vortex stretching would be desirable, as would inclusion of condensation, evaporation, and inertial effects [18]. These improvements, all of which involve significant computational complexity and expense, can be expected to alter detailed properties, and might well change the exponent values. But

since chaotic advection is a generic feature of atmospheric models, one should expect scale-invariant distributions to appear. This is supported by the robustness of the power laws found in simulations, to variations in number and strength of the vortices. In this regard it is also interesting to note that simulations of turbulent MHD processes reproduce power-law burst distributions for solar flares [19], and that tracer patterns similar to those reported here are also found in simulations of two-dimensional barotropic turbulence [20]. Theoretical prediction of the rain and drought distributions, and of the associated exponents, starting from a model velocity field, remains as a formidable challenge.

In summary, I find that tracer distributions in two-dimensional flow, represented by a system of point vortices, exhibit scale invariance during the early stage of the dispersal process. It therefore seems reasonable to attribute power-law rain and drought distributions to turbulent atmospheric flow, and to develop more realistic models, to understand the observations in greater detail.

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REFERENCES

- [1] O. Peters, C. Hertlein, and K. Christensen, Phys. Rev. Lett. **88**, 018701 (2002).
- [2] O. Peters and K. Christensen, Phys. Rev. E **66**, 036120 (2002).
- [3] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 3411 (1967).
- [4] J. M. Carlson, J. S. Langer, and B. E. Shaw, Rev. Mod. Phys. **66**, 657 (1994).
- [5] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
- [6] W. D. McComb, *The Physics of Fluid Turbulence* (Oxford University Press, Oxford, 1990).
- [7] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
- [8] D. J. Tritton, *Physical Fluid Dynamics* (Oxford University Press, Oxford, 1988).
- [9] T. Elperin et al., Phys. Rev. E **66**, 036302 (2002).
- [10] H. Aref, J. Fluid Mech. **143**, 1 (1984).
- [11] See Chs. 12 and 13 of [6] and references therein.
- [12] G. Falkovich, K. Gawędzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).
- [13] A. Sommerfeld, *Mechanics of Deformable Bodies* (Academic Press, New York, 1950).
- [14] H. Aref and E. D. Siggia, J. Fluid Mech. **100**, 705 (1980); E. D. Siggia and H. Aref, Phys. Fluids **24**, 171 (1981).
- [15] R. H. Kraichnan and D. Montgomery, Rep. Prog. Phys. **43**, 35 (1980); R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
- [16] D. G. Dritschel and B. Legras, Phys. Today, March, 1993, p. 44, and references therein.
- [17] Further details on the simulations will be given in R. Dickman, unpublished.
- [18] I. J. Benczik, Z. Toroczkai, and T. Tél, Phys. Rev. Lett. **89**, 164501 (2002).
- [19] G. Boffetta et al., Phys. Rev. Lett. **83**, 4662 (1999).
- [20] G. Haller and G. Yuan, Physica D **147**, 352 (2000).

FIGURE CAPTIONS

FIG. 1. Positions of 10^4 tracers at time $\simeq 1.3\tau_C$, in a system with 126 vortices, $\alpha = 3$. The tracers are initially distributed uniformly over the small square near the right edge. Inset: detail of a region of high tracer density, with two vortices.

FIG. 2. Rain size distribution in a study with 126 vortices, $\alpha = 2$, and $T \simeq \tau_c/2$. The straight line has slope -0.95. Inset: stretched exponential rain size distribution for $T \simeq 200\tau_C$, $\alpha = 4$.

FIG. 3. Drought-duration (main graph) and rain-duration (inset) distributions for the same parameters as Fig. 2. The straight line has slope -1.14.

FIG. 4. Rain-size (main graph) and drought-duration (inset) distributions in a system of 10 vortices of equal strength, $T \simeq 0.85\tau_c$. The straight lines have slopes of -1.01 (rain size) and -1.13 (drought).







